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Multipole in an external electromagnetic field

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Abstract. In this paper, using a distribution theory, a useful formula to obtain the most general classical relativistic equation of motion for an arbitrary multipole point particle, in an external electromagnetic field, in Minkowski space, is derived. The formula is formally obtained from the singular solutions of Maxwell's equations and the definition of the energy-momentum tensor neglecting radiation reaction. As a particular application of this formula, we extract the general equation of motion for a point particle possessing up to quadrupole moments.

1. Introduction

As is well known, the equation of motion of a point charge in an external electromagnetic field, without considering self-field effects, is governed by the Lorentz force. On the other hand, for a point charge endowed with a magnetic moment there is controversy about its equation of motion (see e.g. [1], p 357, and [2]). Evidently the situation is more complex when we wish to discuss the problem of how a general multipole point particle moves through an external electromagnetic field.

There are some papers in the literature (e.g. [3–6]) concerned directly or indirectly with this problem. However, in spite of these works, the fundamental question of what can be extracted from Maxwell's equation and local conservation laws, has not been answered in full. The main reason is perhaps due to the non-integrable singularities of the electromagnetic field produced by the multipole. These singularities are not a problem for the case of a point charge (in the external field approach). However, they are an obstacle for other multipole moments. In order to avoid these singularities, additional prescriptions, obscuring the relation with the Maxwell's equations, are introduced (see e.g. the discussion for the 'simple' case of a charge–dipole particle in [7]). Among these prescriptions, we should mention the basic relation of charged matter moving in an electromagnetic field, $F^{\mu\nu}$, that is,

$$\partial_\nu \theta^{\mu\nu}_{\rm elm} = -F^{\mu\alpha} j_\alpha \qquad (1.1)$$

where $\theta^{\mu\nu}_{\rm elm}$ is the electromagnetic energy-momentum tensor and j_α the charge–current four-vector.

Equation (1.1) is a postulate since it has not been proved for point charge-current models. Additionally, (1.1) must be supplemented with a mathematical meaning since j_α is not a mathematical function of the space-time points. Usually (1.1) is represented as a derivative in the distribution sense in several ways. However, this point of view

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is inconsistent even in the approximation of an external field, that is, when $F^{\mu\nu} \simeq F^{\mu\nu}_{\text{ext}}$. In fact the divergence of $\theta^{\mu\nu}_{\text{elm}}$ cannot be established before defining $\theta^{\mu\nu}_{\text{elm}}$. We have to define $\theta^{\mu\nu}_{\text{elm}}$ first and then derive (1.1) or its rigorous equivalent.

The purpose of this paper is to show, without taking into account self-interaction effects and from as general a viewpoint as possible, what can be extracted from Maxwell's equations and local conservation laws, for multipole point particles, when the electromagnetic field and the energy-momentum tensor, as well as the charge current, are well defined distributions. Using a distribution theory, we have been able to extract neatly the information contained in the singularities of the field. This direct treatment of these singularities is the main difference between our approach and previous results.

As our main result, we shall obtain a formula, equation (3.9), which allows us to give a straightforward derivation of the most general equation of motion for an arbitrary multipole point particle in an external electromagnetic field. As we shall see, this formula determines in which sense (1.1) can be extracted from Maxwell's equations. Accordingly, we regard the question of giving an equivalent of (1.1), for point particles in the external field approach, as having been settled in section 3.

A particular application of (3.9), for the case of a particle endowed with charge, dipole and quadrupole moments, is explicitly considered. Part of the obtained equations agree with earlier theories except for the explicit obtaining of the dynamical contribution of the external electromagnetic field to the 'internal' energy-momentum properties of the multipole particle. In earlier theories these terms do not appear for the general multipole, even though they are well known in the dipole case. For example, there exists the magnetodynamic effect which has momentum associated with the vector product of the external electric field and the magnetic moment [8].

Throughout this paper the metric tensor will have signature +2, and the speed of light is taken as 1. When convenient, indices on vectors and tensors will be omitted and scalar products will be indicated by a dot. Parentheses (...) and brackets [...] will denote total symmetrization and antisymmetrization respectively, of the enclosed indices [5, 9]. The multipole world line (MWL) is $z(\tau)$, where τ is the proper time, $v(\tau) \equiv v (v^2 = -1)$ and $a(\tau) \equiv a$ $(v \cdot a = 0)$ are the four-velocity and four-acceleration respectively. Retarded coordinates will be used in our calculations (see e.g. [10] and [11]). Accordingly, for any space-time point x, we define $R \equiv x - z(\tau_r)$, $R \cdot R = 0$ $(R^0 > 0)$, $\rho \equiv -v \cdot R$, $u \equiv R/\rho - v$, τ_r being the value of the proper time on the intersection between the light cone, with the apex at x opening into the past, and the MWL.

By Ω, we denote an arbitrary open bounded connected region of the space-time that contains at most a connected segment of the MWL. The set of test functions consists of all infinitely differentiable complex-valued functions having compact support in Ω [12, 13]. Topologized in the usual manner [13], this set is denoted by $\mathfrak{D}(\Omega)$. A distribution is a continuous linear functional on $\mathfrak{D}(\Omega)$. A tensor distribution is considered as a family of several distributions (one distribution for given indices). By $\mathfrak{C} \subseteq \mathfrak{D}(\Omega)$, we mean the linear subspace of all $\phi \in \mathfrak{D}(\Omega)$, such that

$$(\theta^{\mu\nu}_{\text{elm}}, \phi) \equiv \int \theta^{\mu\nu}_{\text{elm}}(x)\phi(x)\, \mathrm{d}^4x \tag{1.2}$$

exists. Here

$$\theta^{\mu\nu}_{\text{elm}} \equiv \frac{1}{4\pi}(F^{\mu\alpha}F^{\nu}_{\ \alpha} - \tfrac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}). \tag{1.3}$$

By $\mathfrak{C}_m \subset \mathfrak{D}(\Omega)$, $m=0,1,2,\ldots$, we denote the set of functions in $\mathfrak{D}(\Omega)$ such that all the derivatives, of order up to and including m, vanish at the MWL included in Ω.

A point particle is characterized by the following distributions [5]:

$$j^{\mu} \equiv \sum_{n=0}^{N} (-1)^n \int \mathrm{d}\tau\, m^{\alpha_1\ldots\alpha_n\mu}(\tau)\partial_{\alpha_1}\ldots\partial_{\alpha_n}\delta[x-z(\tau)] \tag{1.4}$$

$$K^{\mu\nu} \equiv \sum_{n=0}^{M} \int \mathrm{d}\tau\, \Gamma^{\mu\nu\alpha_1\ldots\alpha_n}(\tau)\partial_{\alpha_1}\ldots\partial_{\alpha_n}\delta[x-z(\tau)] \tag{1.5}$$

which represent the charge-current distribution and the particle energy-momentum tensor concentrated in the MWL respectively. By definition:

$$(j^{\mu},\phi) \equiv \sum_{n=0}^{N} \int \mathrm{d}\tau\, m^{\alpha_1\ldots\alpha_n\mu}(\tau)\partial_{\alpha_1}\ldots\partial_{\alpha_n}\phi[z(\tau)] \qquad \forall\phi\in\mathfrak{D}(\Omega) \tag{1.6}$$

where $\partial_{\alpha_1}\ldots\partial_{\alpha_n}\phi[z(\tau)] \equiv \partial_{\alpha_1}\ldots\partial_{\alpha_n}\phi[x]|_{x=z(\tau)}$. $K^{\mu\nu}$ is defined analogously. The current multipole moment of order n has the following properties [5, 14]:

$$m^{\alpha_1\ldots\alpha_{n-1}\alpha_n\mu} = m^{(\alpha_1\ldots\alpha_{n-1})[\alpha_n,\mu]} \tag{1.7}$$

$$v_{\alpha_1} m^{\alpha_1\ldots\alpha_n\mu} = 0 \qquad n\geq 2 \tag{1.8}$$

$$m^{\alpha_1\ldots\alpha_{n-1}[\alpha_n\mu\nu]} = 0 \qquad n\geq 1. \tag{1.9}$$

The current multipole of order zero is $m^{\mu} \equiv ev^{\mu}$, with e the total charge of the particle. That is, m^{μ} describes the monopole structure of the point source. The current multipole of order 1, $m^{[\alpha,\mu]}$, describes the electromagnetic dipole and so forth.

Without loss of generality, we suppose that the energy multiple of order n has the following property:

$$v_{\alpha_1}\Gamma^{\mu\nu\alpha_1\ldots\alpha_n} = 0 \qquad n\geq 1. \tag{1.10}$$

This paper is organized as follows. In section 2, the general form of the distribution definition of the energy-momentum tensor for the multipole point particle in the external field is developed. In section 3, the divergence of the electromagnetic energy-momentum tensor is calculated. In section 4, the general form of the equation of motion for a point particle possessing monopole, dipole and quadrupole charge-current moments is derived. Finally, section 5 is devoted to concluding comments.

2. Distribution definition of the energy-momentum tensor

The aim of this section is to discuss the distribution form of the total energy-momentum tensor for our multipole point particle and the electromagnetic field.

The electromagnetic field is considered as a continuous linear functional from $\mathfrak{D}(\Omega)$ into the set of skew-symmetric complex tensors of rank 2, which satisfies Maxwell's equations:

$$(\partial_{\nu}F^{\mu\nu},\phi) = 4\pi(j^{\mu},\phi) \qquad \forall\phi\in\mathfrak{D}(\Omega) \tag{2.1}$$

$$(\partial^{[\alpha}F^{\lambda\gamma]},\phi) = 0 \qquad \forall\phi\in\mathfrak{D}(\Omega) \tag{2.2}$$

where j^{μ} is the vector distribution defined by (1.6).

The electromagnetic field can be decomposed as

$$F = F_{\text{ext}} + F_{\text{ret}} \tag{2.3}$$

where the non-singular part, F_{ext}, satisfies Maxwell's equation for vacuum and F_{ret} is the retarded distribution solution of (2.1) and (2.2). Introducing the four-potential distribution A^α defined by

$$2(A^{[\alpha}, \partial^{\beta]}\phi) \equiv (F_{\text{ret}}^{\alpha\beta}, \phi) \qquad \forall\phi \in \mathfrak{D}(\Omega) \tag{2.4}$$

we obtain, in the Lorentz gauge, that A^α is given by

$$(A^\alpha, \phi) = (\mathscr{A}^{\alpha_1\cdots\alpha_n\alpha}, \partial_{\alpha_1}\ldots\partial_{\alpha_n}\phi) \qquad \forall\phi \in \mathfrak{D}(\Omega) \tag{2.5}$$

where

$$(\mathscr{A}^{\alpha_1\ldots\alpha_n\alpha}, \phi) \equiv \int \frac{m^{\alpha_1\ldots\alpha_n\alpha}}{\rho}\,\phi\,\mathrm{d}^4x \qquad \forall\phi \in \mathfrak{D}(\Omega). \tag{2.6}$$

It is straightforward to verify, replacing (2.6) in (2.5) and then in (2.4), that F_{ret} given by (2.4) is a particular solution of Maxwell's equations (2.1) and (2.2). This gives, formally, the usual solutions found in the literature.

Following Poincaré [15] and von Laue [16], the total energy-momentum tensor is written as

$$\theta^{\mu\nu} \equiv K^{\mu\nu} + \theta_{\text{elm}}^{\mu\nu} \tag{2.7}$$

where $K^{\mu\nu}$ is given by (1.5), and $\theta_{\text{elm}}^{\mu\nu}$ is well defined on $\mathfrak{C}$ through (1.2) and (1.3). Taking into account the superposition shown in (2.3), we obtain (in an obvious notation) the following splitting:

$$\theta_{\text{elm}} = \theta_{\text{ext}} + \theta_{\text{mix}} + \theta_{\text{ret}}. \tag{2.8}$$

Since F_{ext} is not singular, θ_{ext} defines a regular distribution (through (1.2)). On the other hand, except for the monopole case, θ_{mix} is singular and does not define a regular distribution. Let us recall that, even though the singularities of θ_{ret} are stronger, we shall not take them into account since we do not consider self-interaction effects (external field approach). In consequence the singularities of θ_{mix} will be our only concern.

As long as $\phi \in \mathfrak{C}$, θ_{mix} is a well defined functional. Then, our problem is to extend this functional on to the whole space, $\mathfrak{D}(\Omega)$, in such a manner that its extensions are distributions, and to determine the degree of arbitrariness of such extensions. A similar task was achieved in [17] for the monopole case taking self-forces into account. We shall call these extensions 'renormalizations'. Let us now look for the 'renormalizations' of θ_{mix}.

Note that θ_{mix} defines a regular functional in $\mathfrak{C}_{N-1}$ (N, $N \geq 1$, is defined by (1.4)), that is

$$(\theta_{\text{mix}}^{\mu\nu}, \phi) \equiv \int \theta_{\text{mix}}^{\mu\nu}(x)\phi(x)\,\mathrm{d}^4x \qquad \forall\phi \in \mathfrak{C}_{N-1}. \tag{2.9}$$

It is easy to prove that the set $\mathfrak{C}$, for θ_{mix}, is $\mathfrak{C}_{N-1}$. Then, choosing (2.9) as our starting-point, we are not changing the theory where it is well defined.

Now, we can obtain an extension of (2.9) for $\phi \in \mathfrak{D}(\Omega)$. In fact, using the equivalent of (2.4) and (2.5) in functional form ($\rho \neq 0$), the functional definition of $\theta_{\rm mix}$ ($\rho \neq 0$) and integrating by parts, (2.9) can also be written as

$$\begin{aligned}4\pi(\theta^{\mu\nu}_{\rm mix}, \phi) = 4\pi(\theta^{\mu\nu}_{\rm mix}[e], \phi) + \sum_{n=1}^{N} \Bigg\{ \int & \mathrm{d}^4x \mathscr{B}_{\alpha_2 \ldots \alpha_n \alpha} \partial^{\alpha_n} \ldots \partial^{\alpha_2}[F^{\nu\alpha}_{\rm ext} \partial^{\mu} \phi] \\ & - \int \mathrm{d}^4x\, \mathscr{B}_{\alpha_2 \ldots \alpha_n}{}^{\mu} \partial^{\alpha_n} \ldots \partial^{\alpha_2}[F^{\nu\alpha}_{\rm ext} \partial_{\alpha} \phi] \\ & + \int \mathrm{d}^4x\, \mathscr{B}_{\alpha_2 \ldots \alpha_n \alpha} \partial^{\alpha_n} \ldots \partial^{\alpha_2}[F^{\mu\alpha}_{\rm ext} \partial^{\nu} \phi] \\ & - \int \mathrm{d}^4x\, \mathscr{B}_{\alpha_2 \ldots \alpha_n}{}^{\nu} \partial^{\alpha_n} \ldots \partial^{\alpha_2}[F^{\mu\alpha}_{\rm ext} \partial_{\alpha} \phi] \\ & - \int \mathrm{d}^4x\, \mathscr{B}_{\alpha_2 \ldots \alpha_n \alpha} \partial^{\alpha_n} \ldots \partial^{\alpha_2}[g^{\mu\nu} F^{\gamma\alpha}_{\rm ext} \partial_{\gamma} \phi] \\ & + \int \mathrm{d}^4x\, \mathscr{B}_{\alpha_2 \ldots \alpha_n \alpha} \partial^{\alpha_n} \ldots \partial^{\alpha_2}[(\partial^{\mu} F^{\nu\alpha}_{\rm ext} + \partial^{\nu} F^{\mu\alpha}_{\rm ext}) \phi] \Bigg\} \\ & \forall \phi \in \mathfrak{C}_{N-1} \end{aligned} \tag{2.10}$$

where

$$\mathscr{B}^{\alpha_2 \ldots \alpha_n \alpha} \equiv \partial_{\alpha_1} \mathscr{A}^{\alpha_1 \alpha_2 \ldots \alpha_n \alpha} \equiv \partial_{\alpha_1} \left\{ \frac{m^{\alpha_1 \ldots \alpha_n \alpha}}{\rho} \right\} \qquad \rho \neq 0 \tag{2.11}$$

and $\theta_{\rm mix}[e]$ is the corresponding contribution of the monopole. In (2.11) $\mathscr{A}$ is defined as a function whenever $\rho \neq 0$. Since $\mathscr{B}$ is a locally integrable function (by hypothesis $F_{\rm ext}$ is a function defined in $\bar{\Omega}$ such that it has continuous derivatives of order up to and including $N+1$), the right-hand side of (2.10) defines a linear and continuous functional on $\mathfrak{D}(\Omega)$. Therefore, (2.10) provides one of the renormalizations that should be sought.

We shall write the distribution obtained through (2.10), which is not $\theta_{\rm mix}[e]$, as $\theta_{\rm reg}$. Specifically, we define

$$(\theta_{\rm mix}, \phi) \equiv (\theta_{\rm mix}[e], \phi) + (\theta_{\rm reg}, \phi) \qquad \forall \phi \in \mathfrak{D}(\Omega) \tag{2.12}$$

with the left-hand side interpreted as in (2.10).

Obviously, $\theta_{\rm mix}(e) + \theta_{\rm reg}$ is not the unique extension of (2.9). If $\theta_{\rm ren}$ is another renormalization, then necessarily,

$$(\theta_{\rm ren} - \theta_{\rm mix}[e] - \theta_{\rm reg}, \phi) = 0 \qquad \forall \phi \in \mathfrak{C}_{N-1}. \tag{2.13}$$

That is, the distribution $\theta_{\rm ren} - \theta_{\rm mix}[e] - \theta_{\rm reg}$ vanishes on $\mathfrak{C}_{N-1}$. In other words, the order and the support of $\theta_{\rm ren} - \theta_{\rm mix}[e] - \theta_{\rm reg}$ are at most $N-1$ and the MWL respectively.

Thus, *the most general renormalization of* (2.9) is

$$\theta_{\rm ren} = \theta_{\rm mix}[e] + \theta_{\rm reg} + \Lambda \tag{2.14}$$

where Λ is a distribution of order at most $N-1$, in Ω, having support at the MWL.

For physical reasons, distributions whose supports consist of isolated points will not be considered. Consequently, the general form of Λ is of the type given in (1.5) with M at most equal to $N-1$.

From these considerations, we conclude that the general form of θ can be written as

$$\theta = K + \theta_{\text{mix}}[e] + \theta_{\text{reg}} + \theta_{\text{ext}} \tag{2.15}$$

where the arbitrariness in the renormalization of θ_{mix}, Λ, has been left to the material energy-momentum tensor K. This is always possible because Λ has finite order and support at the MWL.

This splitting of θ in a part concentrated at the MWL, K, and the other part, θ_{elm}, is the physical one. The part concentrated at the MWL should be properly called the particle energy-momentum tensor (the contribution of the field in K is considered as part of the particle). On the other hand, θ_{elm} represents the contribution to θ of the pure Maxwell's field. As a matter of fact, θ_{elm} is the only part of θ which vanishes as the external field goes to zero. Therefore, it is physically unreasonable to consider any other mathematically valid extension of θ_{elm}.

Equation (2.15) provides us with the general form of the energy-momentum tensor distribution that describes the interaction between the multipole and the external electromagnetic field.

3. Divergence of the electromagnetic energy-momentum tensor

In this section we wish to calculate the non-trivial part of the divergence of the energy-momentum tensor, that is, the divergence of θ_{elm}. From this divergence, as we shall see in section 4, the equation of motion for an arbitrary multipole is easily obtained. For this reason we present the calculation in some detail.

From (2.10) and (2.12), we have

$$\begin{aligned} 4\pi(\partial_\nu\theta^{\mu\nu}_{\text{reg}}, \phi) &= -4\pi(\theta^{\mu\nu}_{\text{reg}}, \partial_\nu\phi) \\ &= \sum_{n=1}^{N}\left\{-\int d^4x\, \mathscr{B}_{\alpha_2\ldots\alpha_n\alpha}\partial^{\alpha_n}\ldots\partial^{\alpha_2}[F^{\mu\alpha}_{\text{ext}}\partial^\nu\partial_\nu\phi]\right. \\ &\quad -\int d^4x\, \mathscr{B}_{\alpha_2\ldots\alpha_n\alpha}\partial^{\alpha_n}\ldots\partial^{\alpha_2}[(\partial^\mu F^{\nu\alpha}_{\text{ext}}+\partial^\nu F^{\mu\alpha}_{\text{ext}})\partial_\nu\phi] \\ &\quad \left.+\int d^4x\, \mathscr{B}_{\alpha_2\ldots\alpha_n}{}^{\nu}\partial^{\alpha_n}\ldots\partial^{\alpha_2}[F^{\mu\alpha}_{\text{ext}}\partial_\alpha\partial_\nu\phi]\right\} \qquad \forall\phi\in\mathfrak{D}(\Omega). \end{aligned} \tag{3.1}$$

Using $\partial^\mu F^{\nu\alpha}_{\text{ext}} = \partial^\nu F^{\mu\alpha}_{\text{ext}} - \partial^\alpha F^{\mu\nu}_{\text{ext}}$, $\partial\cdot\partial F^{\mu\alpha}_{\text{ext}} = 0$, and $\mathscr{B}_{\alpha_2\ldots\alpha_n\alpha} \equiv \mathscr{B}_{\alpha_2\ldots[\alpha_n\alpha]}$, it follows that

$$4\pi(\partial_\nu\theta^{\mu\nu}_{\text{reg}}, \phi) = -\sum_{n=1}^{N}\int d^4x\, \mathscr{B}_{\alpha_2\ldots\alpha_n\alpha}\partial^{\alpha_n}\ldots\partial^{\alpha_2}\partial\cdot\partial[F^{\mu\alpha}_{\text{ext}}\phi] \qquad \forall\phi\in\mathfrak{D}(\Omega). \tag{3.2}$$

Let us consider the region, of the space-time $\bar{\Omega}(\varepsilon)$, defined by the inequalities $\tau_1 \leqslant \tau_r \leqslant \tau_2$ and $\varepsilon \leqslant \rho \leqslant \sup_\Omega \rho$, $\varepsilon \geqslant 0$. Since Ω is bounded, we can choose τ_1 and τ_2 such that $\Omega \subset \bar{\Omega}(o)$. Taking into account that ϕ and all its derivatives vanish identically outside Ω, we may assume that the region of integration in (3.2) is $\bar{\Omega}(o)$. Therefore, we can rewrite (3.2) as

$$-4\pi(\partial_\nu\theta^{\mu\nu}_{\text{reg}}, \phi) = \sum_{n=1}^{N}\lim_{\varepsilon\to 0}\int_{\bar{\Omega}(\varepsilon)} d^4x\, \mathscr{B}_{\alpha_2\ldots\alpha_n\alpha}\partial^{\alpha_n}\ldots\partial^{\alpha_2}\partial\cdot\partial[F^{\mu\alpha}_{\text{ext}}\phi] \qquad \forall\phi\in\mathfrak{D}(\Omega) \tag{3.3}$$

which follows because of the Lebesgue-dominated convergence theorem. This form is more adequate to obtain the formula we are looking for.

Integrating by parts, and using in $\bar{\Omega}(\varepsilon)$, $\partial\cdot\partial\mathscr{B}_{\alpha_2\ldots\alpha_n\alpha}=0$, it follows that

$$
\begin{aligned}
-\int_{\bar{\Omega}(\varepsilon)} \mathrm{d}^4x\, \mathscr{B}_{\alpha_2\ldots\alpha_n\alpha}\partial^{\alpha_n}\ldots\partial^{\alpha_2}\partial\cdot\partial[F^{\mu\alpha}_{\text{ext}}\phi]
&= \int_{B(\varepsilon)} \mathrm{d}B_\nu\, \mathscr{B}_{\alpha_2\ldots\alpha_n\alpha}\partial^{\alpha_n}\ldots\partial^{\alpha_2}\partial^\nu[F^{\mu\alpha}_{\text{ext}}\phi] \\
&\quad + \int_{\bar{\Omega}(\varepsilon)} \mathrm{d}^4x\, \partial_\nu\mathscr{B}_{\alpha_2\ldots\alpha_n\alpha}\partial^{\alpha_n}\ldots\partial^{\alpha_2}\partial^\nu[F^{\mu\alpha}_{\text{ext}}\phi] \\
&= \int_{B(\varepsilon)} \mathrm{d}B_\nu\, \mathscr{B}_{\alpha_2\ldots\alpha_n\alpha}\partial^{\alpha_n}\ldots\partial^{\alpha_2}\partial^\nu[F^{\mu\alpha}_{\text{ext}}\phi] \\
&\quad - \int_{B(\varepsilon)} \mathrm{d}B_\nu\, \partial^\nu\mathscr{B}_{\alpha_2\ldots\alpha_n\alpha}\partial^{\alpha_n}\ldots\partial^{\alpha_2}[F^{\mu\alpha}_{\text{ext}}\phi]
\end{aligned}
\tag{3.4}
$$

where $\mathrm{d}B_\nu$ is the surface element of the segment of the Bhabha tube $\rho=\varepsilon$.

Let us define, in $\bar{\Omega}(\varepsilon)$, the following tensor:

$$
\mathscr{C}^{[\nu\alpha_1]\alpha_2\ldots\alpha_n\alpha}\equiv\partial^\nu\mathscr{A}^{\alpha_1\alpha_2\ldots\alpha_n\alpha}-\partial^{\alpha_1}\mathscr{A}^{\nu\alpha_2\ldots\alpha_n\alpha}.
\tag{3.5}
$$

Since, in $\bar{\Omega}(\varepsilon)$, $\partial\cdot\partial\mathscr{A}^{\alpha_1\alpha_2\ldots\alpha_n\alpha}=0$, we have

$$
\partial_\sigma\mathscr{C}^{[\nu\sigma]\alpha_2\ldots\alpha_n\alpha}=\partial^\nu\partial_{\alpha_1}\mathscr{A}^{\alpha_1\alpha_2\ldots\alpha_n\alpha}=\partial^\nu\mathscr{B}^{\alpha_2\ldots\alpha_n\alpha}.
\tag{3.6}
$$

Now, replacing (3.6) in (3.4) and applying Stokes's theorem to the region $B(\varepsilon)$, we obtain, from (3.3),

$$
\begin{aligned}
4\pi(\partial_\nu\theta^{\mu\nu}_{\text{reg}},\phi)=\sum_{n=1}^{N}\lim_{\varepsilon\to0}\bigg\{&\int_{B(\varepsilon)}\mathrm{d}B_\nu\,\mathscr{B}_{\alpha_2\ldots\alpha_n\alpha}\partial^{\alpha_n}\ldots\partial^{\alpha_2}\partial^\nu[F^{\mu\alpha}_{\text{ext}}\phi] \\
&+\int_{B(\varepsilon)}\mathrm{d}B_\nu\,\mathscr{C}^{[\nu\sigma]}{}_{\alpha_2\ldots\alpha_n\alpha}\partial_\sigma\partial^{\alpha_n}\ldots\partial^{\alpha_2}[F^{\mu\alpha}_{\text{ext}}\phi]\bigg\}.
\end{aligned}
\tag{3.7}
$$

Using definition (2.11), every limit in (3.7) exists and is easy to calculate. Carrying out these calculations, (3.7) reduces to

$$
(\partial_\nu\theta^{\mu\nu}_{\text{reg}},\phi)=-\sum_{n=1}^{N}\int\mathrm{d}\tau\,m^{\alpha_1\ldots\alpha_n}{}_\alpha(\tau)\partial_{\alpha_1}\ldots\partial_{\alpha_n}\{F^{\mu\alpha}_{\text{ext}}\phi\}[z(\tau)]\qquad\forall\phi\in\mathfrak{D}(\Omega).
\tag{3.8}
$$

From (2.7), (2.15), (3.8) and the well known result for the divergence of $\theta_{\text{mix}}[e]$, we finally obtain

$$
(\partial_\nu\theta^{\mu\nu}_{\text{elm}},\phi)=-\sum_{n=0}^{N}\int\mathrm{d}\tau\,m^{\alpha_1\ldots\alpha_n}{}_\alpha(\tau)\partial_{\alpha_1}\ldots\partial_{\alpha_n}\{F^{\mu\alpha}_{\text{ext}}\phi\}[z(\tau)]\qquad\forall\phi\in\mathscr{D}(\Omega).
\tag{3.9}
$$

We wish to conclude this section with two remarks. First, it is important to emphasize that (3.9) follows from the distribution definition of θ_{elm} which is the only part of θ that becomes zero when the external field vanishes. Second, if F_{ext} is an infinitely differentiable function, (3.9) can be rewritten as

$$
(\partial_\nu\theta^{\mu\nu}_{\text{elm}},\phi)=-(j_\alpha,F^{\mu\alpha}_{\text{ext}}\phi)=-(F^{\mu\alpha}_{\text{ext}}j_\alpha,\phi)\qquad\forall\phi\in\mathfrak{D}(\Omega).
\tag{3.10}
$$

Equation (3.10) tells us how the continuity of physical description is kept when the field and the energy-momentum tensor are well defined distributions. In this sense this result is consistent with the interpretation of $\theta_{\rm elm}$ given in section 2. We note that (3.10) is not defined in general, but we can use (3.9) to define the distribution $F^{\mu\alpha}_{\rm ext} j_\alpha$, giving a meaning to (1.1) for point multipoles with $\theta_{\rm elm}$ well defined in the external field approach.

4. Equations of motion

In this section we derive the general form of the equation of motion for the case of a particle endowed with monopole, dipole and quadrupole electromagnetic moments. This particular case exhibits all the complications of the general situation of an arbitrary particle with several multipole moments. The equation of motion is extracted imposing local energy-momentum and angular momentum conservations.

Let us define the tensor

$$M^{\lambda\mu\nu} \equiv x^{[\lambda}\theta^{\mu]\nu}. \tag{4.1}$$

Equation (4.1) is a well defined distribution because the multipliers x are infinitely differentiable.

Let us demand

(i) energy-momentum conservation, that is

$$(\partial_\nu \theta^{\mu\nu}, \phi) = 0 \tag{4.2}$$

(ii) angular momentum conservation, that is,

$$(\partial_\nu M^{\lambda\mu\nu}, \phi) = 0 \tag{4.3}$$

(iii) the index M [see (1.5)] should have the lowest possible value consistent with (i) and (ii).

Condition (iii) means that we choose the simplest K that is compatible with local conservation laws.

It can be shown that $M = 0$ is incompatible with (4.2) and the quadrupolar structure. We note that, in the external field approach, (4.2) and (4.3) are compatible with a monopolar or dipolar structure for $M = 0$. Then, the conservation laws impose a structure of 'spin' for the quadrupole but not necessarily for the dipole. Of course, this is valid only in the external field approach. The structure of the particle must be more complicated if the radiation reaction is taken into account (for instance, even in the monopole case the conservation laws are not satisfied for $M = 0$ [17]).

Therefore, the next order to consider is $M = 1$. In this case, (1.5) gives

$$K^{\mu\nu} = \int \mathrm{d}\tau \{\Gamma^{\mu\nu}(\tau) + \Gamma^{\mu\nu\lambda}(\tau)\partial_\lambda\}\delta[x - z(\tau)]. \tag{4.4}$$

In order to obtain information from (4.2) and (4.3) let us break down the energy multipole moments $\Gamma^{\mu\nu}$ and $\Gamma^{\mu\nu\lambda}$ into the forms

$$\Gamma^{\mu\nu} = m_0 v^\mu v^\nu + m^\mu v^\nu + n^\nu v^\mu + n^{\mu\nu} \tag{4.5}$$

$$\Gamma^{\mu\nu\lambda} = h^\lambda v^\mu v^\nu + h^{\mu\lambda} v^\nu + q^{\nu\lambda} v^\nu + h^{\mu\nu\lambda} \tag{4.6}$$

where each index that does not label a v labels a quantity orthogonal to v. This decomposition leads to the notion of space and time parts of $\Gamma^{\mu\nu}$ and $\Gamma^{\mu\nu\lambda}$. Similarly, we can decompose the charge-current quadrupole moment as

$$m^{\alpha_1\alpha_2\alpha} \equiv 2\mathscr{E}^{\alpha_1[\alpha_2}v^{\alpha]} + \mathscr{M}^{\alpha_1\alpha_2\alpha} \tag{4.7}$$

where

$$\mathscr{E}^{\mu\nu} \equiv -m^{\mu\nu\alpha}v_\alpha. \tag{4.8}$$

It follows [5], that $\mathscr{E}$ is symmetric, $\mathscr{M}$ has the symmetries (1.7) and (1.9) of m, and $\mathscr{E}$ and $\mathscr{M}$ are orthogonal to v on all indices. $\mathscr{E}$ and $\mathscr{M}$ may be called the electric and magnetic parts of the charge-current quadrupole m.

Replacing (4.5) and (4.6) in (4.4), integrating by parts and using (3.9), (4.7) and (4.8), (4.2) may be written

$$\begin{aligned}\int \mathrm{d}\tau\Bigg\{\Bigg[&\frac{\mathrm{d}}{\mathrm{d}\tau}(m_0v^\mu + m^\mu - v^\mu\dot{h}\cdot v - \dot{h}^{\mu\alpha}v_\alpha + F^{\mu\alpha}_{\text{ext}}m_{\alpha\nu}v^\nu - \mathscr{E}^{\alpha\nu}\partial_\alpha F^{\mu}_{\text{ext}\,\nu} - \dot{\mathscr{E}}^{\beta}_{\alpha}v_\beta F^{\mu\alpha}_{\text{ext}})\\ &- eF^{\mu\nu}_{\text{ext}}v_\nu - m^{\alpha\nu}\partial_\alpha F^{\mu}_{\text{ext}\,\nu} - m^{\alpha\nu\lambda}\partial_\alpha\partial_\nu F^{\mu}_{\text{ext}\,\lambda}\Bigg]\phi[z(\tau)]\\ &-[v^\mu n^\beta + n^{\mu\beta} + v^\mu\dot{h}^\beta_\perp + a^\mu h^\beta + \dot{h}^{\mu\beta}_\perp - F^{\mu}_{\text{ext}\,\alpha}m^{\alpha\beta}_\perp + \dot{\mathscr{E}}^{\beta\alpha}_\perp F^{\mu}_{\text{ext}\,\alpha}\\ &+ 2\mathscr{M}^{(\alpha\beta)\nu}\partial_\alpha F^{\mu}_{\text{ext}\,\nu} + 2\mathscr{E}^{\alpha\beta}v^\nu\partial_\alpha F^{\mu}_{\text{ext}\,\nu}]\partial_\beta\phi[z(\tau)]\\ &+[v^\mu q^{(\alpha\nu)} + h^{\mu(\alpha\nu)} - \mathscr{E}^{\alpha\nu}F^{\mu\lambda}_{\text{ext}}v_\lambda - \mathscr{M}^{(\alpha\nu)\lambda}F^{\mu}_{\text{ext}\lambda}]\partial_\alpha\partial_\nu\phi[z(\tau)]\Bigg\} = 0\end{aligned} \tag{4.9}$$

where

$$\frac{\mathrm{d}h^\mu}{\mathrm{d}\tau} \equiv \dot{h}^\mu \equiv -(\dot{h}\cdot v)v^\mu + \dot{h}^\mu_\perp \tag{4.10}$$

$$\frac{\mathrm{d}h^{\mu\lambda}}{\mathrm{d}\tau} \equiv \dot{h}^{\mu\lambda} \equiv -\dot{h}^{\mu\alpha}v_\alpha v^\lambda + \dot{h}^{\mu\lambda}_\perp \tag{4.11}$$

$$m^{\alpha\beta} \equiv -m^{\alpha\mu}v_\mu v^\beta + m^{\alpha\beta}_\perp \tag{4.12}$$

$$\frac{\mathrm{d}\mathscr{E}^{\beta\alpha}}{\mathrm{d}\tau} \equiv \dot{\mathscr{E}}^{\beta\alpha} \equiv -\dot{\mathscr{E}}^{\mu\alpha}v_\mu v^\beta + \dot{\mathscr{E}}^{\beta\alpha}_\perp. \tag{4.13}$$

Following similar steps, equation (4.3) may be written

$$\begin{aligned}\int \mathrm{d}\tau\{&[m^\mu v^\lambda + n^\lambda v^\mu + n^{\mu\lambda} - m^\lambda v^\mu - n^\mu v^\lambda - n^{\lambda\mu}]\phi(z(\tau)]\\ &-[h^{\mu\beta}v^\lambda + q^{\lambda\beta}v^\mu + h^{\mu\lambda\beta} - h^{\lambda\beta}v^\mu - q^{\mu\beta}v^\lambda - h^{\lambda\mu\beta}]\partial_\beta\phi[z(\tau)]\} = 0.\end{aligned} \tag{4.14}$$

From (4.9) and (4.14), we obtain (cf [17])

$$q^{(\mu\nu)} = -\mathscr{M}^{(\mu\nu)}{}_\alpha F^{\lambda\alpha}_{\text{ext}}v_\lambda \tag{4.15}$$

$$h^{\alpha(\mu\nu)} = \mathscr{E}^{\mu\nu}F^{\alpha\lambda}_{\text{ext}}v_\lambda + \mathscr{M}^{(\mu\nu)}{}_\lambda F^{\alpha\lambda}_{\text{ext}} + v^\alpha\mathscr{M}^{(\mu\nu)}_\beta F^{\lambda\beta}_{\text{ext}}v_\lambda \tag{4.16}$$

$$h^{\mu\beta} = q^{\mu\beta} \tag{4.17}$$

$$h^{[\mu\lambda]\beta} = 0 \tag{4.18}$$

$$\begin{aligned}m^\beta = n^\beta = &-\dot{h}^\beta_\perp + \dot{q}^{\mu\beta}v_\mu - v_\mu F^{\mu}_{\text{ext}\,\alpha}m^{\alpha\beta}_\perp + v_\mu\dot{q}^{\mu\alpha}v_\alpha v^\beta\\ &+ 2\mathscr{M}^{(\lambda\beta)\alpha}v_\mu\partial_\lambda F^{\mu}_{\text{ext}\alpha} + \dot{\mathscr{E}}^{\beta\alpha}_\perp F^{\mu}_{\text{ext}\,\alpha}v_\mu\end{aligned} \tag{4.19}$$

$$n^{\mu\beta} = -\dot{q}^{(\mu\beta)} - 2v^{(\mu}\dot{q}_s^{\beta)\alpha}v_\alpha - v^\mu v^\beta v_\lambda \dot{q}^{\lambda\alpha}v_\alpha - h^{(\mu}a^{\beta)}$$
$$+ v_\lambda F^\lambda_{\text{ext}\,\alpha} m_\perp^{\alpha(\beta}v^{\mu)} + m_\perp^{\alpha(\beta}F^{\mu)}_{\text{ext}\,\alpha} - 2v^{(\mu}\mathcal{M}_s^{\beta)\nu\alpha}v_\lambda\partial_\nu F^\lambda_{\text{ext}\,\alpha}$$
$$-2\mathcal{M}^{\nu(\beta}{}_{s\alpha}\partial_\nu F^{\mu)\alpha}_{\text{ext}} - v^{(\mu}\dot{\mathcal{E}}_\perp^{\beta)\alpha}F^\lambda_{\text{ext}\,\alpha}v_\lambda - F^{(\mu}_{\text{ext}\,\alpha}\dot{\mathcal{E}}_\perp^{\beta)\alpha} - 2\mathcal{E}^{\nu(\beta}\partial_\nu F^{\mu)\alpha}_{\text{ext}}v_\alpha \quad (4.20)$$

$$\dot{q}^{[\mu\beta]} - 2v^{[\mu}\dot{q}_A^{\beta]\alpha}v_\alpha + a^{[\mu}h^{\beta]} = v_\lambda F^\lambda_{\text{ext}\,\alpha} m_\perp^{\alpha[\beta}v^{\mu]} + m_\perp^{\alpha[\beta}F^{\mu]}_{\text{ext}\,\alpha}$$
$$-2v^{[\mu}\mathcal{M}_s^{\beta]\nu\alpha}v_\lambda\partial_\nu F^\lambda_{\text{ext}\,\alpha} - 2\partial_\nu F^{[\mu}_{\text{ext}\,\alpha}\mathcal{M}_s^{\beta]\nu\alpha} - v^{[\mu}\dot{\mathcal{E}}_\perp^{\beta]\alpha}F^\lambda_{\text{ext}\,\alpha}v_\lambda$$
$$-F^{[\mu}_{\text{ext}\,\alpha}\dot{\mathcal{E}}_\perp^{\beta]\alpha} - 2\mathcal{E}^{\nu[\beta}\partial_\nu F^{\mu]\alpha}_{\text{ext}}v_\alpha \quad (4.21)$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\{m_0 v^\mu - \dot{h}^\mu - 2v^\mu \dot{h}\cdot v - 2\dot{q}^{[\mu\lambda]}v_\lambda - v_\lambda F^\lambda_{\text{ext}\,\alpha}m_\perp^{\alpha\mu} + 2\mathcal{M}^{(\nu\mu)\alpha}v_\lambda\partial_\nu F^\lambda_{\text{ext}\,\alpha}$$
$$-\mathcal{E}^{\nu\alpha}\partial_\nu F^\mu_{\text{ext}\,\alpha} + \dot{\mathcal{E}}_\perp^{\mu\alpha}v_\lambda F^\lambda_{\text{ext}\,\alpha} + F^{\mu\alpha}_{\text{ext}}m_{\alpha\nu}v^\nu - \dot{\mathcal{E}}^{\beta\alpha}v_\beta F^\mu_{\text{ext}\,\alpha}\}$$
$$= eF^{\mu\nu}_{\text{ext}}v_\nu + m^{\alpha\nu}\partial_\alpha F^\mu_{\text{ext}\,\nu} + m^{\alpha\nu\lambda}\partial_\alpha\partial_\nu F^\mu_{\text{ext}\,\lambda} \quad (4.22)$$

where

$$q_s^{\mu\nu} \equiv q^{(\mu\nu)} \qquad q_A^{\mu\nu} \equiv q^{[\mu\nu]} \quad (4.23)$$

$$\mathcal{M}_s^{\mu\nu\lambda} \equiv \mathcal{M}^{(\mu\nu)\lambda}. \quad (4.24)$$

From (4.15) to (4.22), we see that the external electromagnetic field contributes dynamically to the 'internal' energy-momentum properties of the multipole particle. In earlier theories, for the quadrupole case, these terms do not appear explicitly. For another multipole, these dynamical contributions of the external field can be analogously obtained from (3.9) and the conservation laws (4.2) and (4.3).

In order to compare part of our equations with previous findings, let us define the following tensors:

$$S^{\mu\nu} \equiv 2q^{[\mu\nu]} - 2h^{[\mu}v^{\nu]} \quad (4.25)$$

$$p^\mu \equiv mv^\mu - \dot{S}^{\mu\nu}v_\nu \quad (4.26)$$

$$m \equiv m_0 - \dot{h}\cdot v. \quad (4.27)$$

Then, in terms of $S^{\mu\nu}$ and p^μ, (4.21) and (4.22) are written

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(S^{\mu\nu} + S^{\mu\nu}_{\text{ext}}) = 2p^{[\mu}v^{\nu]} + 2p^{[\mu}_{\text{ext}}v^{\nu]} + D^{\mu\nu} \quad (4.28)$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(p^\mu + p^\mu_{\text{ext}}) = \sum_{n=0}^{2} m^{\alpha_1\ldots\alpha_n\alpha}(\tau)\partial_{\alpha_1}\ldots\partial_{\alpha_n}F^\mu_{\text{ext}\,\alpha} \quad (4.29)$$

where

$$p^\mu_{\text{ext}} \equiv -v_\lambda F^\lambda_{\text{ext}\,\alpha}m_\perp^{\alpha\mu} + 2\mathcal{M}^{(\nu\mu)\alpha}v_\lambda\partial_\nu F^\lambda_{\text{ext}\,\alpha} - \mathcal{E}^{\nu\alpha}\partial_\nu F^\mu_{\text{ext}\,\alpha}$$
$$+ \dot{\mathcal{E}}_\perp^{\mu\alpha}v_\lambda F^\lambda_{\text{ext}\,\alpha} + F^{\mu\alpha}_{\text{ext}}m_{\alpha\nu}v^\nu - \dot{\mathcal{E}}^{\beta\alpha}v_\beta F^\mu_{\text{ext}\,\alpha} \quad (4.30)$$

$$S^{\mu\nu}_{\text{ext}} \equiv 2F^{[\mu}_{\text{ext}\,\alpha}\mathcal{E}^{\nu]\alpha} \quad (4.31)$$

$$D^{\mu\nu} \equiv -2F^{[\mu}_{\text{ext}\,\alpha}m^{\nu]\alpha} - 4\partial_\lambda F^{[\mu}_{\text{ext}\,\alpha}m_s^{\nu]\lambda\alpha}$$
$$m_s^{\nu\lambda\alpha} \equiv m^{(\nu\lambda)\alpha}. \quad (4.32)$$

If we include $S^{\mu\nu}_{\rm ext}$ and $p^{\mu}_{\rm ext}$ in a new definition of spin and momentum of the particle, (4.28) and (4.29) coincide with earlier theories [3–6] when they are specified for the pole-dipole-quadrupole case. In this manner, earlier theories in the external field approach are dealt within equations (4.28) and (4.29).

We conclude this section by pointing out that (4.28) and (4.29) need to be completed with additional equations in order to define uniquely the motion of the particle. For instance, we can use the following state equation:

$$m^{\mu\nu} = \Lambda S^{\mu\nu} \tag{4.33}$$

which gives a relativistic generalization of the well known proportionality between the angular momentum vector and the magnetic moment vector. These supplementary conditions, corresponding to the final definition of the model, are outside the scope of Maxwell's equations and the local laws (4.2) and (4.3).

5. Discussion

We have shown that the strictly multipole point particle model can be completely incorporated with Maxwell's theory, in the external field approach, in a theory that satisfies unambiguously local balance laws of energy and angular momentum. Equation (3.9) supplies a firm basis for the covariant formulation of the electrodynamics of a multipole point particle in an external electromagnetic field.

One could argue that the external field approach should be derived from the equations including radiation effects instead of neglecting radiation reaction from the beginning. However, it is important to observe that inclusion of the radiation reaction contributions changes the equations of motion only in quadratic terms of the relevant parameters of the diverse multipoles (such parameters are: the magnitude of charge, the magnitude of magnetic dipole and so on, cf for example [6] and [11]). On the other hand, the usual external field approach is a linear theory in the diverse multipoles. It is then a trivial matter to see that obtaining the external field approach by neglecting products between these characteristic parameters in the full equations is equivalent to neglecting $\theta_{\rm ret}$ from the beginning.

From our results and from considerations of the radiation reaction problem (see [11] and [17]), and in spite of the partial success in the treatment of singularities in previous theories, we can conclude that with a distribution theory the classical electrodynamics of point models can be viewed in a new context. In fact the distribution theory establishes a coherent and precise framework in which the essential of point particles may be worked out more clearly. Also, it allows us to understand from a fundamental and unified standpoint that which had been hidden by other arguments (e.g. particular cut-off procedures).

From (3.9), and using the conservation laws (4.2) and (4.3), we see that in order to satisfy local balance laws the multipole particle must have a minimal structure. This 'material' structure grows as the order of the electromagnetic multipole grows. This situation is explicit in the calculations in section 4. Accordingly, if the radiation reaction is taken into account, it is evident that a similar relationship should hold true, the only difference being that the electromagnetic self-field renormalizes more quantities, thus giving additional structure to the multipole particle.

Equations (4.15)–(4.22) are a list of properties that a point particle possessing up to quadrupole moments should satisfy in the external field approach. Thus, what we have obtained in section 4 is a framework encompassing several kinds of quadrupolar particles rather than the theory of any particular quadrupole subject to its particular equations. The problem of considering the conditions that lead to a uniquely defined world line is obviously physically important. Indeed, it is related to the physical meaning of the quantities which appear in the equation of motion. All these considerations correspond to point out specific multipole models which need to be thoroughly investigated. For an arbitrary multipole point particle, some possible restrictions have been discussed in [18].

It could be useful to recapitulate the difference between our approach and some previous ideas. As we have already mentioned, other formulations of multipole point particles (see e.g. [3], [4], [6], [18] and [19]) when specified in the external field approximation, differ from our work in the treatment of $\theta_{\rm elm}$. Indeed, [3], [4], [6], [18] and [19] do not have a theory for $\theta_{\rm elm}$ at the MWL. Nevertheless, these references extract the value of the divergence of $\theta_{\rm elm}$ through prescriptions. These prescriptions (cut-off procedures, postulates) impose restrictions on the force and torque acting on the particle without clarifying its relationship with Maxwell's equations. Also, the prescriptions on $\partial\cdot\theta_{\rm elm}$ lead to the corresponding ones on $\partial\cdot K$ (otherwise, there could be inconsistencies). On the other hand, our approach consists in defining $\theta_{\rm elm}$ from the solutions of Maxwell's equations, as a consequence we extract its divergence [given by (3.9)], and from it we ask for the simplest K such that $\partial\cdot(K+\theta_{\rm elm})=0$. This difference in treatment leads to different results (see, e.g. for the quadrupole case, the term $S^{\mu\nu}_{\rm ext}$ in (4.28)).

In the comparison of our results with those obtained by Dixon [5] for an extended charged body, we observe that he concludes that the only relations between the multipole moments imposed by the 'generalized conservation equations' are the conservation of charge and the equations of motion for the four momentum and the spin tensor. On the other hand, for the strictly point model the situation seems completely different. In fact it is clear from section 4 that the consideration of multipole higher than quadrupole leads, even in the external field approach, to additional equations of motion for the energy-momentum multipoles that follow the spin. Nevertheless, we could arrange the additional equations in such a manner that they are automatically satisfied (as state equations) and recover Dixon's statement. If this can be reasonably carried out, it is our opinion that this is one of the several possibilities of definitive models and not the only one for the strictly point multipole.

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